Technical Companion to

A Generalization of Linear Positive Systems with Applications to Nonlinear Systems: Invariant Sets and the Poincaré–Bendixson Property

A Self-Contained Guide for EE/Math Undergraduates

Companion to the paper by Eyal Weiss and Michael Margaliot
(Automatica, 2021)

Abstract

This document explains every equation, definition, and theorem in the paper *"A Generalization of Linear Positive Systems with Applications to Nonlinear Systems"* in detail, assuming only undergraduate-level linear algebra and ordinary differential equations. No prior knowledge of positive systems, k-positive systems, compound matrices, totally positive matrices, or the Poincaré–Bendixson property is required. Each equation is accompanied by (1) a plain-English description of what it says, (2) a symbol-by-symbol breakdown, (3) an intuitive explanation of why it matters, and (4) concrete numerical examples where helpful.

Contents

1 Background: Positive Systems, Cones, and Sign Variations

1.1 What Is a Positive Dynamical System?

In many areas of science and engineering, we encounter systems whose state variables represent inherently nonnegative quantities: populations in biology, chemical concentrations in reaction networks, probabilities in Markov chains, or the density of particles in compartmental models. A **positive dynamical system** is one whose dynamics *preserve nonnegativity*: if every state variable starts nonnegative, it remains nonnegative for all future time.

Formally, consider the linear time-varying (LTV) system

$$\dot{x}(\tau) = A(\tau)\, x(\tau), \quad x(t_0) = x_0, \tag{1}$$

where $x(\tau) \in \mathbb{R}^n$ is the state vector and $A(\tau) \in \mathbb{R}^{n\times n}$ is a matrix that may change over time.

Symbol-by-symbol breakdown:

- $x(\tau) \in \mathbb{R}^n$: the state at time τ. Each entry $x_i(\tau)$ is one state variable (e.g., a concentration, a population count).
- $\dot{x}(\tau) := \frac{d}{d\tau}x(\tau)$: the time derivative of the state.
- $A(\tau) \in \mathbb{R}^{n\times n}$: the **system matrix** at time τ. It determines how the current state drives the rates of change.
- $x(t_0) = x_0$: the **initial condition**—the state at the starting time t_0.

The system is called **positive** on a time interval (a, b) if for every pair (t_0, t) with $a < t_0 < t < b$ and any $x(t_0) \in \mathbb{R}^n_+$, we have $x(t) \in \mathbb{R}^n_+$, where

$$\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x_i \geq 0 \text{ for all } i\}$$

is the **nonnegative orthant**.

Key Idea

A positive system maps the "all nonnegative" cone to itself. If you start with all state variables nonnegative, they stay nonnegative forever. This is the most natural property for systems modeling inherently nonnegative quantities.

A Simple Positive System (n

Consider $\dot{x} = Ax$ with

$$A = \begin{bmatrix} -1 & 2 \\ 3 & -4 \end{bmatrix}.$$

Starting from $x(0) = [1,\ 0]'$:

- $\dot{x}_1(0) = -1 \cdot 1 + 2 \cdot 0 = -1$ (decreasing),
- $\dot{x}_2(0) = 3 \cdot 1 + (-4) \cdot 0 = 3$ (increasing).

Even though x_1 is decreasing, note that $x_2 = 0$ and $\dot{x}_2 = 3 > 0$, so x_2 is moving *into* the positive region. The off-diagonal entries ($a_{12} = 2 \geq 0$ and $a_{21} = 3 \geq 0$) ensure that whenever one variable approaches zero, the other variables "push it back up." This is the hallmark of a positive system.

1.2 The Transition Matrix

The solution of (1) can be written as $x(t) = \Phi(t, t_0)\, x(t_0)$, where $\Phi(t, t_0)$ is the **transition matrix** (also called the *state-transition matrix* or *fundamental matrix*). It satisfies:

$$\dot{\Phi}(\tau) = A(\tau)\, \Phi(\tau), \quad \Phi(t_0) = I. \tag{2}$$

What it says in plain English: The transition matrix $\Phi(t, t_0)$ encodes *all possible responses* of the system. Given any initial condition x_0, the state at any later time t is simply $\Phi(t, t_0)\, x_0$—a matrix-vector multiplication.

Symbol breakdown:

- $\Phi(t, t_0) \in \mathbb{R}^{n \times n}$: a square matrix. Its (i, j) entry tells you how much the j-th initial condition contributes to the i-th state variable at time t.
- $\Phi(t_0) = I$: at the starting time, the transition matrix is the identity. This makes sense: if you ask "where is the system at the time it started?" the answer is "exactly where it started."

Positivity Through the Transition Matrix

The LTV system (1) is positive on (a, b) if and only if $\Phi(t, t_0) \geq 0$ (every entry is nonnegative) for all $a < t_0 < t < b$. In turn, this holds if and only if $A(\tau)$ is **Metzler** for all $a < \tau < b$.

1.3 Metzler Matrices

A matrix $A \in \mathbb{R}^{n \times n}$ is called **Metzler** if every off-diagonal entry is nonnegative:

$$a_{ij} \geq 0 \quad \text{for all } i \neq j.$$

The diagonal entries can be anything (positive, negative, or zero). Only the off-diagonal entries must be nonnegative.

Metzler vs. Non-Metzler

$$\underbrace{\begin{bmatrix} -3 & 2 \\ 1 & -5 \end{bmatrix}}_{\text{Metzler}} \qquad \underbrace{\begin{bmatrix} -3 & 2 \\ -1 & -5 \end{bmatrix}}_{\text{NOT Metzler } (a_{21}=-1<0)}$$

Why Metzler matrices make systems positive: Consider $\dot{x} = Ax$ and suppose $x_i = 0$ while all other $x_j \geq 0$. Then $\dot{x}_i = \sum_j a_{ij} x_j = a_{ii} \cdot 0 + \sum_{j \neq i} a_{ij} x_j$. If A is Metzler, every $a_{ij} \geq 0$ for $j \neq i$, and $x_j \geq 0$, so $\dot{x}_i \geq 0$. The i-th variable cannot decrease below zero—it is "pushed back" whenever it reaches the boundary.

1.4 The Number of Sign Variations in a Vector

The central concept that enables the generalization in this paper is the **number of sign variations** in a vector. Intuitively, this counts how many times the entries of a vector "change sign" as you scan from the first entry to the last.

For a vector $y \in \mathbb{R}^n$ with *no zero entries*, the number of sign variations is:

$$\sigma(y) := |\{i \in \{1, \ldots, n-1\} : y_i\, y_{i+1} < 0\}|.$$

What it computes: Count the number of consecutive pairs (y_i, y_{i+1}) where one is positive and the other is negative. Each such pair represents a "sign change."

Counting Sign Variations

For $y = \begin{bmatrix} -4.2 & 3 & -0.5 \end{bmatrix}'$:

- Pair $(y_1, y_2) = (-4.2, 3)$: different signs $\to$ sign change.
- Pair $(y_2, y_3) = (3, -0.5)$: different signs $\to$ sign change.

So $\sigma(y) = 2$.
For $y = \begin{bmatrix} 1 & 3 & 2 & -1 \end{bmatrix}'$:

- $(1, 3)$: same sign. $(3, 2)$: same sign. $(2, -1)$: different signs.

So $\sigma(y) = 1$.
For $y = \begin{bmatrix} 5 & 2 & 7 & 1 \end{bmatrix}'$: all entries positive, so $\sigma(y) = 0$.

1.5 Handling Zero Entries: s^- and s^+

When a vector contains zero entries, the definition of "sign variation" becomes ambiguous—a zero entry could be thought of as either positive or negative. The paper uses two standard definitions from the theory of totally positive matrices:

The "minimum" count $s^-(y)$: For $y \in \mathbb{R}^n$, if $y = 0$ then $s^-(y) = 0$. Otherwise, $s^-(y) := \sigma(\bar{y})$, where $\bar{y}$ is obtained from y by *deleting all zero entries*. In other words, s^- counts sign variations while *ignoring* all zeros.

The "maximum" count $s^+(y)$: $s^+(y) := \max_{z \in S(y)} \sigma(z)$, where $S(y)$ is the set of all vectors obtained by replacing every zero entry of y by either $+1$ or -1. In other words, s^+ counts sign variations while *choosing the worst-case replacement* for each zero.

Computing s^- and s^+

Let $n = 5$ and $y = \begin{bmatrix} -1 & 1 & 0 & 0 & -3.5 \end{bmatrix}'$.
Computing $s^-(y)$: Delete zeros to get $\bar{y} = \begin{bmatrix} -1 & 1 & -3.5 \end{bmatrix}'$. Sign changes: $(-1, 1)$ is a change, $(1, -3.5)$ is a change. So $s^-(y) = \sigma(\bar{y}) = 2$.
Computing $s^+(y)$: Replace each zero by ± 1 to maximize sign changes. The best choice is to make the zeros alternate in sign to create as many changes as possible. Try $y^* = \begin{bmatrix} -1 & 1 & -1 & 1 & -3.5 \end{bmatrix}'$: sign changes at every consecutive pair, so $\sigma(y^*) = 4$. Thus $s^+(y) = 4$.

Key Idea

$s^-(y) \le s^+(y)$ always. s^- gives the most optimistic (fewest) count of sign variations. s^+ gives the most pessimistic (most) count. For vectors with no zero entries, $s^-(y) = s^+(y) = \sigma(y)$.

The paper notes the bounds:

$$0 \le s^-(y) \le s^+(y) \le n - 1 \quad \text{for all } y \in \mathbb{R}^n.$$

1.6 The Duality Relation Between s^- and s^+

Let $D := \operatorname{diag}(1, -1, 1, \ldots, (-1)^{n-1})$. Then for all $x \in \mathbb{R}^n$:

$$s^-(x) + s^+(Dx) = n - 1. \tag{8}$$

What this says: If you flip the sign of every other entry (multiply by D), the s^+ count of the modified vector and the s^- count of the original vector always sum to $n-1$. This is a "conservation law" for sign variations.

The Duality Relation

Let $n = 5$ and $y = \begin{bmatrix}-1 & 1 & 0 & 0 & -3.5\end{bmatrix}'$. We computed $s^-(y) = 2$ above.
Now $D = \operatorname{diag}(1, -1, 1, -1, 1)$, so $Dy = \begin{bmatrix}-1 & -1 & 0 & 0 & -3.5\end{bmatrix}'$. $s^+(Dy)$: replace zeros to maximize sign variations. Try $\begin{bmatrix}-1 & -1 & 1 & -1 & -3.5\end{bmatrix}'$: changes at positions 2–3 and 3–4, giving $\sigma = 2$. Try $\begin{bmatrix}-1 & -1 & 1 & 1 & -3.5\end{bmatrix}'$: changes at 2–3 and 4–5, giving $\sigma = 2$. Try $\begin{bmatrix}-1 & -1 & -1 & 1 & -3.5\end{bmatrix}'$: changes at 3–4 and 4–5, giving $\sigma = 2$. The maximum is $s^+(Dy) = 2$.
Check: $s^-(y) + s^+(Dy) = 2 + 2 = 4 = n - 1$. ✓

1.7 The Set $\mathcal{V}$ of "Clean" Vectors

The paper defines a special set of vectors where s^- and s^+ agree:

$$\mathcal{V} := \{x \in \mathbb{R}^n : s^-(x) = s^+(x)\}.$$

A vector x belongs to $\mathcal{V}$ if and only if:

$$x_1 \neq 0, \; x_n \neq 0, \; \text{and if } x_i = 0 \text{ for some } i \in [2, n-1], \; \text{then } x_{i-1}\, x_{i+1} < 0. \tag{7}$$

What this means: The first and last entries must be nonzero, and any interior zero must be "sandwiched" between entries of opposite sign. Under these conditions, the zero does not create ambiguity—it is clear that a sign change occurs at that position regardless of how you replace the zero.

Numerical Example

The vector $x = \begin{bmatrix}1 & 0 & -1\end{bmatrix}'$ is in $\mathcal{V}$: $x_1 = 1 \neq 0$, $x_3 = -1 \neq 0$, and the zero at position 2 is sandwiched between entries of opposite sign ($1 \cdot (-1) = -1 < 0$).
The vector $x = \begin{bmatrix}1 & 0 & 1\end{bmatrix}'$ is **not** in $\mathcal{V}$: the zero is sandwiched between entries of the *same* sign ($1 \cdot 1 = 1 > 0$). Here $s^-(x) = 0$ but $s^+(x) = 2$ (replace the zero by -1).

The importance of $\mathcal{V}$ will become clear in Section 5: strongly k-positive systems drive trajectories into $\mathcal{V}$, where the number of sign variations becomes unambiguous.

2 Sign-Regular Matrices and the Variation Diminishing Property

2.1 What Is a Sign-Regular Matrix?

The paper builds on the classical theory of **totally nonnegative (TN)** and **totally positive (TP)** matrices. These are matrices whose *minors* (determinants of square submatrices) have special sign properties.

Definition 2.1 (Sign-regular matrices)**.** A matrix $A \in \mathbb{R}^{n \times m}$ is:

- **sign-regular of order** k (SR_k) if all its minors of order k are either all nonnegative or all nonpositive.
- **strictly sign-regular of order** k (SSR_k) if it is SR_k, all minors of order k are nonzero, and they all have the same sign.

- **sign-regular** (SR) if it is SR_k for all k.
- **strictly sign-regular** (SSR) if it is SSR_k for all k.

Special cases:

- If all entries of A are nonnegative, then A is SR_1 (all 1×1 minors = entries ≥ 0).
- A **totally nonnegative (TN)** matrix has all minors of all orders nonnegative.
- A **totally positive (TP)** matrix has all minors of all orders strictly positive.
- TP $\implies$ SSR.

Checking Sign-Regularity

Consider $A = \begin{bmatrix} 1 & 1/4 \\ 40 & 2 \end{bmatrix}$.
1×1 minors (entries): $1, 1/4, 40, 2$ — all positive. So A is SSR_1.
2×2 minor: $\det(A) = 1 \cdot 2 - (1/4) \cdot 40 = 2 - 10 = -8$. The single 2×2 minor is negative (nonzero, all have the same sign trivially), so A is SSR_2.
Since A is SSR_k for all $k \in \{1, 2\}$, it is SSR.
Note: A is *not* totally nonnegative because $\det(A) = -8 < 0$.

2.2 The Variation Diminishing Property (VDP)

The key property of TN and TP matrices that the paper exploits is:

Variation Diminishing Property

If A is totally nonnegative (TN), then

$$s^-(Ax) \leq s^-(x) \quad \text{for all } x \in \mathbb{R}^m.$$

If A is totally positive (TP), then

$$s^+(Ax) \leq s^-(x) \quad \text{for all } x \in \mathbb{R}^m \setminus \{0\}.$$

Multiplying a vector by a TN/TP matrix can only *decrease* (or maintain) the number of sign variations—it can never increase them.

Why this matters for dynamical systems: In a positive system, the transition matrix $\Phi(t, t_0)$ is nonnegative, which means $s^-(x) = 0$ is preserved (nonnegative vectors stay nonnegative). The paper's key insight is to generalize this: if $\Phi(t, t_0)$ has the variation diminishing property, then the number of sign variations in $x(t)$ cannot increase over time. This is the foundation for k-positive systems.

The connection between SSR_k and the VDP is made precise by:

Theorem 2.1 (Theorem 1 in the paper)**.** *Let $A \in \mathbb{R}^{n \times n}$ be nonsingular. Pick $k \in [1, n]$. The following are equivalent:*

(a) For any $x \in \mathbb{R}^n \setminus \{0\}$ with $s^-(x) \leq k - 1$, we have $s^+(Ax) \leq k - 1$.
(b) A is SSR_k.

What it says in plain English: A nonsingular matrix A maps every vector with at most $k-1$ sign changes (in the s^- sense) to a vector with at most $k-1$ sign changes (in the s^+ sense) if and only if A is strictly sign-regular of order k.

Theorem 1 for k

When $k = 1$, the condition becomes: for any nonzero x with $s^-(x) = 0$ (i.e., all entries are the same sign), Ax satisfies $s^+(Ax) = 0$ (i.e., all entries of Ax are the same sign). This is equivalent to saying that A is SSR_1, meaning all entries of A are either all positive or all negative.

Similarly, there is an analogous result for SR_k matrices (Theorem 2 in the paper):

Theorem 2.2 (Theorem 2 in the paper)**.** *Let $A \in \mathbb{R}^{n\times n}$ be nonsingular. Pick $k \in [1, n]$. The following are equivalent:*

(a) For any $x \in \mathbb{R}^n$ with $s^-(x) \le k-1$, we have $s^-(Ax) \le k-1$.
(b) A is SR_k.

3 Compound Matrices: The Engine of the Theory

3.1 Minors and the Multiplicative Compound

Given a matrix $A \in \mathbb{R}^{n\times n}$ and an integer $k \in [1, n]$, a **minor of order** k is the determinant of a $k \times k$ submatrix formed by selecting k rows and k columns.

Each minor is specified by a set of row indices $\alpha = \{i_1, \dots, i_k\}$ with $1 \le i_1 < \cdots < i_k \le n$ and column indices $\beta = \{j_1, \dots, j_k\}$ with $1 \le j_1 < \cdots < j_k \le n$. The minor is denoted $A(\alpha|\beta)$.

Computing a Minor

Let $A = \begin{bmatrix} 4 & 5 & 6 \\ -1 & 4 & -2 \\ 0 & 3 & -3 \end{bmatrix}$.
With $\alpha = \{1, 3\}$ and $\beta = \{2, 3\}$:

$$A(\alpha|\beta) = \det \begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix} = \det \begin{bmatrix} 5 & 6 \\ 3 & -3 \end{bmatrix} = 5 \cdot (-3) - 6 \cdot 3 = -15 - 18 = -33.$$

The k**th multiplicative compound matrix** $A^{(k)}$ is the $\binom{n}{k} \times \binom{n}{k}$ matrix whose entries are all the minors of order k of A, arranged in lexicographic order.

For example, when $n = 3$ and $k = 2$, the row/column index pairs are $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$ (in lexicographic order), giving:

$$A^{(2)} = \begin{bmatrix} A(\{1,2\}|\{1,2\}) & A(\{1,2\}|\{1,3\}) & A(\{1,2\}|\{2,3\}) \\ A(\{1,3\}|\{1,2\}) & A(\{1,3\}|\{1,3\}) & A(\{1,3\}|\{2,3\}) \\ A(\{2,3\}|\{1,2\}) & A(\{2,3\}|\{1,3\}) & A(\{2,3\}|\{2,3\}) \end{bmatrix}.$$

Note the special cases: $A^{(1)} = A$ and $A^{(n)} = \det(A)$.

The Cauchy–Binet Formula

The Cauchy–Binet formula gives $(AB)^{(k)} = A^{(k)} B^{(k)}$. This means the kth compound of a product is the product of the kth compounds. This multiplicative property is crucial: it says that the minors of a transition matrix can be tracked using the compound matrix.

The Size of Compound Matrices

For an $n \times n$ matrix and compound order k, the compound matrix is $\binom{n}{k} \times \binom{n}{k}$. For $n = 10$ and $k = 5$, this is 252×252—much larger than the original matrix! The paper's results avoid computing compound matrices of the transition matrix by instead working with additive compounds of the system matrix A, which have the same dimension.

3.2 The Additive Compound Matrix $A^{[k]}$

The k**th additive compound** of A is defined as:

$$A^{[k]} := \frac{d}{d\varepsilon}(I + \varepsilon A)^{(k)}\Big|_{\varepsilon=0}.$$

This means: take the identity plus a small perturbation εA, compute its kth multiplicative compound, and differentiate with respect to ε at $\varepsilon = 0$. The result captures how the minors of a matrix respond to infinitesimal perturbations.

From this definition, one can show:

$$(I + \varepsilon A)^{(k)} = I + \varepsilon A^{[k]} + o(\varepsilon). \tag{11}$$

Why does this matter? For a time-varying matrix $Y(t)$ evolving according to $\dot{Y}(t) = A(t)Y(t)$, the kth multiplicative compound $Y^{(k)}(t)$ satisfies:

$$\frac{d}{dt}Y^{(k)}(t) = A^{[k]}(t)\, Y^{(k)}(t), \tag{13}$$

where $A^{[k]}(t) := (A(t))^{[k]}$. This is a key equation: the dynamics of all the kth-order minors of $Y(t)$ are governed by a *linear system* with system matrix $A^{[k]}(t)$.

Why Additive Compounds Are Central

Equation (13) tells us that to track the sign of the minors of the transition matrix $\Phi(t, t_0)$—and hence its sign-regularity properties—we only need to study the linear system with matrix $A^{[k]}(t)$. If $A^{[k]}(t)$ is Metzler, then the minors of order k remain nonnegative, which makes the transition matrix SR_k and gives us the variation diminishing property.

3.3 Explicit Formula for $A^{[k]}$: Lemma 1

Lemma 3.1 (Lemma 1 in the paper)**.** *The entry of $A^{[k]}$ corresponding to $(\alpha|\beta) = (i_1, \ldots, i_k|j_1, \ldots, j_k)$ is:*

- $\sum_{\ell=1}^{k} a_{i_\ell j_\ell}$ *if $i_\ell = j_\ell$ for all $\ell \in [1, k]$ (i.e., $\alpha = \beta$: this is a diagonal entry);*
- $(-1)^{\ell+m} a_{i_\ell j_m}$ *if α and β coincide except for a single index $i_\ell \neq j_m$; and*
- 0*, otherwise.*

What this tells us:

1. **Diagonal entries** of $A^{[k]}$ are sums of k diagonal entries of A.
2. **Off-diagonal entries** are either zero or equal to $\pm a_{ij}$ for some entry of A.
3. Most entries of $A^{[k]}$ are zero—it is a sparse matrix.

The Additive Compound $A^{[2]}$ for n

For a 3×3 matrix $A = \{a_{ij}\}$, using Lemma 1:

$$A^{[2]} = \begin{bmatrix} a_{11} + a_{22} & a_{23} & -a_{13} \\ a_{32} & a_{11} + a_{33} & a_{12} \\ -a_{31} & a_{21} & a_{22} + a_{33} \end{bmatrix}. \tag{12}$$

Let us verify one entry. The row index $\alpha = \{1, 2\}$ and column index $\beta = \{1, 3\}$ differ in one index: α has 2 where β has 3 (at position $\ell = 2$, $m = 2$). The entry is $(-1)^{2+2} a_{i_2 j_2} = (-1)^4 a_{23} = a_{23}$. ✓
The entry for $\alpha = \{1, 2\}$, $\beta = \{2, 3\}$ differs in one index: α has 1 where β has 3 (at position $\ell = 1$, $m = 2$). The entry is $(-1)^{1+2} a_{13} = -a_{13}$. ✓

The Additive Compound $A^{[2]}$ for n

For a 4×4 matrix, $A^{[2]}$ is $\binom{4}{2} \times \binom{4}{2} = 6 \times 6$:

$$A^{[2]} = \begin{bmatrix} a_{11}{+}a_{22} & a_{23} & a_{24} & -a_{13} & -a_{14} & 0 \\ a_{32} & a_{11}{+}a_{33} & a_{34} & a_{12} & 0 & -a_{14} \\ a_{42} & a_{43} & a_{11}{+}a_{44} & 0 & a_{12} & a_{13} \\ -a_{31} & a_{21} & 0 & a_{22}{+}a_{33} & a_{34} & -a_{24} \\ -a_{41} & 0 & a_{21} & a_{43} & a_{22}{+}a_{44} & a_{23} \\ 0 & -a_{41} & a_{31} & -a_{42} & a_{32} & a_{33}{+}a_{44} \end{bmatrix}. \tag{14}$$

Notice the pattern: many zeros, and the nonzero off-diagonal entries are entries of A (possibly with a sign flip).

3.4 When Is $A^{[k]}$ Metzler?

Since the system (13) for the minors is a linear system with matrix $A^{[k]}$, the minors remain nonnegative (i.e., the transition matrix is SR_k) if and only if $A^{[k]}$ is Metzler. This motivates the question: *for a given A, when is $A^{[k]}$ Metzler?*

The paper introduces matrix classes M_k^n to characterize this.

4 The Main Generalization: k-Positive Linear Systems

4.1 The Sets P_-^k and P_+^k

For any $k \in [1, n]$, define:

$$P_-^k := \{z \in \mathbb{R}^n : s^-(z) \le k - 1\}, \qquad P_+^k := \{z \in \mathbb{R}^n : s^+(z) \le k - 1\}.$$

What these sets contain:

- P_-^k is the set of all vectors with at most $k-1$ sign variations (using the optimistic count s^-). It is **closed**.
- P_+^k is the set of all vectors with at most $k-1$ sign variations (using the pessimistic count s^+). It is **open** (as a subset of $\mathbb{R}^n \setminus \{0\}$).
- $P_+^k = \operatorname{int}(P_-^k)$ for all $k \in [1, n-1]$.

Special case $k = 1$:

$$P_-^1 = \mathbb{R}_+^n \cup \mathbb{R}_-^n, \qquad P_+^1 = \operatorname{int} \mathbb{R}_+^n \cup \operatorname{int} \mathbb{R}_-^n, \tag{15}$$

where $\mathbb{R}_-^n := -\mathbb{R}_+^n$. So P_-^1 is the union of the nonnegative and nonpositive orthants—exactly the vectors with zero sign variations.

Nesting property:

$$P_-^1 \subset P_-^2 \subset \cdots \subset P_-^n = \mathbb{R}^n, \qquad P_+^1 \subset P_+^2 \subset \cdots \subset P_+^n = \mathbb{R}^n. \tag{16}$$

The Sets for n

- P_-^1: vectors with $s^-(z) = 0$, i.e., all entries nonneg. or all nonpos. E.g., $[1, 2, 3]'$ or $[-1, -5, 0]'$.
- P_-^2: vectors with $s^-(z) \leq 1$, i.e., at most one sign change. E.g., $[1, 2, -3]'$ (one change at position 2–3) or $[1, 2, 3]'$ (zero changes).
- $P_-^3 = \mathbb{R}^3$: all vectors (since $s^- \leq n - 1 = 2$ always).

The Hierarchy of Sign-Variation Sets

As k increases from 1 to n, the sets P_-^k and P_+^k grow from the nonnegative/nonpositive orthants (most restrictive) to all of $\mathbb{R}^n$ (no restriction). The integer k controls how many sign changes are "allowed."

4.2 Definition of k-Positive Systems

Definition 4.1 (k-positive system)**.** Fix $k \in [1, n]$. The LTV system (1) is **k-positive on** (a, b) if P_-^k is an invariant set of the dynamics, i.e., for any pair $a < t_0 < t < b$ and any $x(t_0) \in P_-^k$, we have $x(t) \in P_-^k$.

What this means in plain English: If the initial state has at most $k-1$ sign variations, then the state at any future time also has at most $k-1$ sign variations. The dynamics cannot *create* new sign variations beyond $k - 1$.

Special case $k = 1$: A 1-positive system maps $P_-^1 = \mathbb{R}_+^n \cup \mathbb{R}_-^n$ to itself. This is exactly a positive system! Nonnegative initial conditions stay nonnegative; nonpositive initial conditions stay nonpositive.

Why this is a genuine generalization: For $k \geq 2$, a system may be k-positive without being positive (1-positive). The system allows up to $k-1$ sign changes in the state vector, and preserves this bound. This is useful because many important systems that are *not* positive (state variables can change sign) still have the property that sign changes cannot proliferate.

A 3-Positive System

Consider the constant-matrix system $\dot{x} = Ax$ with $n = 4$ and

$$A = \begin{bmatrix} -1 & 2 & -2 & 1 \\ 3 & 0 & 1 & -1 \\ -4 & 1.5 & 2 & 4 \\ 1 & -1 & 2 & 5 \end{bmatrix}.$$

Using Lemma 1, one can compute $A^{[3]}$ and verify that it is Metzler (this is Example 6 in the paper):

$$A^{[3]} = \begin{bmatrix} 1 & 4 & 1 & 1 \\ 2 & 4 & 1 & 2 \\ 1 & 1.5 & 6 & 2 \\ 1 & 4 & 3 & 7 \end{bmatrix}.$$

Since all off-diagonal entries of $A^{[3]}$ are nonneg., $A^{[3]}$ is Metzler, so the system is 3-positive. This means: if $x(0)$ has at most 2 sign variations, then $x(t)$ has at most 2 sign variations for all $t \geq 0$.

Note that A itself is *not* Metzler ($a_{13} = -2 < 0$, etc.), so the system is *not* positive (not 1-positive). But it is 3-positive.

The paper shows (Figure 1) that starting from $x(0) = [0.34,\ -0.54,\ -1.06,\ 0.49]'$ with $s^-(x(0)) = 2$, we have $s^-(x(t)) \leq 2$ for all $t \geq 0$.

4.3 The Characterization Theorem

Theorem 4.1 (Theorem 4 in the paper)**.** *The system (1) is k-positive on (a, b) if and only if $A^{[k]}(s)$ is Metzler for almost all $s \in (a, b)$.*

What this says: There is a clean equivalence: k-positivity of the original system $\leftrightarrow$ Metzler property of the kth additive compound of the system matrix. This is powerful because checking whether a matrix is Metzler is straightforward.

Proof sketch:

1. k-positivity means $\Phi(t, t_0)$ is SR_k (by Theorem 2), which means $\Phi^{(k)}(t, t_0) \geq 0$ or ≤ 0.
2. By equation (13), $\Phi^{(k)}$ satisfies $\frac{d}{ds}\Phi^{(k)} = A^{[k]}\Phi^{(k)}$ with $\Phi^{(k)}(t_0) = I$.
3. By continuity, $\Phi^{(k)}(t, t_0)$ must be ≥ 0 (it starts at $I > 0$ and cannot jump to ≤ 0).
4. $\Phi^{(k)}(t, t_0) \geq 0$ for all $t > t_0$ iff $A^{[k]}(s)$ is Metzler for almost all s—this is the standard Metzler characterization for positive LTVs, now applied to the compound system.

Corollary 4.2 (Corollary 1 in the paper)**.** *If the system (1) is 1-positive and 2-positive, then it is k-positive for all $k \in [1, n]$.*

Why: If A and $A^{[2]}$ are both Metzler, a classical result by Schwarz shows that $A^{[k]}$ is Metzler for every k.

5 Strongly k-Positive Systems

5.1 Definition and Motivation

Definition 5.1 (Strongly k-positive)**.** Fix $k \in [1, n]$. The system (1) is **strongly k-positive on** (a, b) if for any pair (t_0, t) with $a < t_0 < t < b$:

$$x(t_0) \in P^k_- \setminus \{0\} \implies x(t) \in P^k_+.$$

What this says: In a (k-positive but not strongly) system, the dynamics map P^k_- to P^k_-. In a *strongly* k-positive system, the dynamics map $P^k_- \setminus \{0\}$ to P^k_+—the *interior*. This is analogous to the difference between a nonnegative matrix and a strictly positive matrix: the latter maps the boundary of the cone into its interior.

Why "strongly" matters: Strong k-positivity ensures that sign variations strictly decrease over time (unless they are already at the minimum possible value). This leads to powerful convergence results.

Theorem 5.1 (Theorem 5 in the paper)**.** *Suppose that $A^{[k]}(s)$ is Metzler for almost all $s \in (a, b)$, and that for any $a < t_0 < t < b$ there exists $t_0 \le \tau < t$ such that $A^{[k]}(s)$ is irreducible at τ^+. Then the system (1) is strongly k-positive on (a, b).*

What this says: If the additive compound is Metzler and is irreducible "frequently enough" (at some point in every subinterval), then the system is strongly k-positive. This is a sufficient condition. For continuous A, a full characterization is:

Theorem 5.2 (Theorem 6 in the paper)**.** *Let $A(\cdot) : (a, b) \to \mathbb{R}^{n\times n}$ be continuous. The system (1) is strongly k-positive on (a, b) iff both:*

1. *$A^{[k]}(\tau)$ is Metzler for all $\tau \in (a, b)$, and*
2. *for any interval $[p, q] \subset (a, b)$, there exists $t^* \in [p, q]$ such that $A^{[k]}(t^*)$ is irreducible.*

The irreducibility condition: A Metzler matrix is irreducible if the directed graph of its nonneg. off-diagonal entries is strongly connected. Intuitively, this means every "component" of the system influences every other component—there are no isolated subsystems. Irreducibility ensures that the system mixes sign variations thoroughly, driving the state into P^k_+ (where the sign variation count is unambiguous).

5.2 The Decreasing Staircase (Proposition 7)

One of the most elegant results in the paper describes how sign variations decrease along trajectories:

Proposition 5.3 (Proposition 7 in the paper)**.** *Assume there exists $k \in [1, n-1]$ such that the system (1) is strongly i-positive for all $i \le k$. Then for any $x(t_0) \in P^k_- \setminus \{0\}$ and any times $t_0 < t_1 < t_2 < \dots$:*

$$s^-(x(t_0)) \ge s^+(x(t_1)) \ge s^-(x(t_1)) \ge s^+(x(t_2)) \ge s^-(x(t_2)) \ge \dots \tag{21}$$

with no more than $k - 1$ strict inequalities. Furthermore, there exists $\tau \ge t_0$ such that $x(t) \in \mathcal{V}$ for all $t \ge \tau$.

What this says in plain English: The number of sign variations forms a "decreasing staircase." Both $s^-(x(t))$ and $s^+(x(t))$ decrease over time, alternating in a nested fashion. After finitely many steps (at most $k-1$ strict decreases), they stabilize and coincide—the trajectory enters $\mathcal{V}$, where the number of sign variations is unambiguous.

Why this is powerful: Both s^- and s^+ are integer-valued Lyapunov functions: they decrease along trajectories and are bounded below by zero. Since they take values in $\{0, 1, \dots, n-1\}$, they can only decrease a finite number of times. This means every trajectory eventually "settles" into a fixed number of sign variations.

The Staircase in Action

Suppose $n = 4$ and the system is strongly i-positive for $i = 1, 2, 3$. Starting from $x(0)$ with $s^-(x(0)) = 2$:

Time	$s^-(x(t))$	$s^+(x(t))$
$t = 0$	2	3
$t = 0.5$	2	2
$t = 1.2$	1	2
$t = 2.0$	1	1
$t \ge 2.0$	1	1

The sign variations decreased from 2–3 down to 1–1 in two steps, after which $s^- = s^+$ (the trajectory is in $\mathcal{V}$) and the count stays at 1 forever. The trajectory has exactly one sign change for all $t \geq 2$: it lives in a subset of $\mathbb{R}^4$ where the entries change sign exactly once.

6 Explicit Algebraic Conditions for k-Positivity

6.1 Overview

Theorem 4 tells us that k-positivity $\Leftrightarrow$ $A^{[k]}$ is Metzler. But checking whether $A^{[k]}$ is Metzler requires computing the additive compound, which is a $\binom{n}{k} \times \binom{n}{k}$ matrix. The paper's main contribution in Section 4 is to translate this into *direct conditions on the entries of* A, avoiding the need to compute $A^{[k]}$.

The answer depends on k:

- $k = 1$: $A^{[1]} = A$, so k-positivity $\Leftrightarrow$ A is Metzler (standard positive system).
- $k = n-1$: $A^{[n-1]}$ is Metzler $\Leftrightarrow$ $A \in M^n_{n-1}$ (competitive system in disguise).
- $k \in [2, n-2]$: $A^{[k]}$ is Metzler $\Leftrightarrow$ $A \in M^n_k$ (a generalized periodic tridiagonal-like structure).

6.2 The Case $k = n - 1$: Competitive Systems

Definition 6.1 (Definition 4 in the paper)**.** M^n_{n-1} is the set of matrices $A \in \mathbb{R}^{n \times n}$ satisfying:

(a) $a_{ij} \geq 0$ for all i, j with $i - j$ odd,
(b) $a_{ij} \leq 0$ for all $i \neq j$ with $i - j$ even.

The Sign Pattern of M^4_3

For $n = 4$, the matrices in M^4_3 have the sign pattern:

$$\begin{bmatrix} * & \geq 0 & \leq 0 & \geq 0 \\ \geq 0 & * & \geq 0 & \leq 0 \\ \leq 0 & \geq 0 & * & \geq 0 \\ \geq 0 & \leq 0 & \geq 0 & * \end{bmatrix},$$

where $*$ denotes "don't care" (the diagonal can be anything). Notice the checkerboard pattern: off-diagonal entries alternate in sign depending on whether $i - j$ is odd or even.

Lemma 6.1 (Lemma 3 in the paper)**.** *$A^{[n-1]}$ is Metzler if and only if $A \in M^n_{n-1}$.*

$(n-1)$-Positive

The paper proves (Lemma 4) that $(n-1)$-positive systems are just competitive systems in disguise. Specifically, if $A \in M^n_{n-1}$, then after a coordinate transformation $y = -DPx$ (where $D = \text{diag}(1, -1, 1, \dots, (-1)^{n-1})$ and P is the "reversal" permutation), the system $\dot{y} = By$ has $(-B)$ Metzler, i.e., the transformed system is competitive.

Recall that a system $\dot{x} = Ax$ is **competitive** if $(-A)$ is Metzler. Competitive systems are dual to cooperative systems: in a competitive system, increasing one state variable tends to decrease the others.

Remark 6.2. This gives a beautiful unification: 1-positive systems are cooperative, $(n-1)$-positive systems are competitive, and k-positive systems for $1 < k < n-1$ interpolate between these two extremes.

6.3 The Case $k = 2$: The Tridiagonal Metzler Class M_2^n

Definition 6.2 (Definition 1 in the paper). M_2^n is the set of matrices $A \in \mathbb{R}^{n\times n}$ satisfying:

(a) $a_{1n}, a_{n1} \leq 0$;
(b) $a_{ij} \geq 0$ for all i, j with $|i - j| = 1$;
(c) $a_{ij} = 0$ for all i, j with $1 < |i - j| < n - 1$.

What this means: The matrix must be **tridiagonal** (all entries more than one step from the diagonal are zero), with nonneg. super- and sub-diagonal entries, and nonpositive corner entries a_{1n} and a_{n1}.

Sign Pattern of M_2^5

For $n = 5$:

$$\begin{bmatrix} * & \geq 0 & 0 & 0 & \leq 0 \\ \geq 0 & * & \geq 0 & 0 & 0 \\ 0 & \geq 0 & * & \geq 0 & 0 \\ 0 & 0 & \geq 0 & * & \geq 0 \\ \leq 0 & 0 & 0 & \geq 0 & * \end{bmatrix}.$$

This is a tridiagonal matrix with nonneg. entries on the super- and sub-diagonals, plus nonpositive entries in the top-right and bottom-left corners.

Lemma 6.3 (Lemma 2 in the paper). *$A^{[2]}$ is Metzler if and only if $A \in M_2^n$.*

Verifying Lemma 2 for n

The additive compound $A^{[2]}$ for $n = 4$ is given in (14). Its off-diagonal entries include: $a_{12}, a_{23}, a_{34}, a_{21}, a_{32}, a_{43}$ (which must be ≥ 0) and $-a_{13}, -a_{14}, -a_{31}, -a_{41}, -a_{24}, -a_{42}$ (which must also be ≥ 0, requiring the original entries to be ≤ 0).
But the conditions $a_{13}, a_{31}, a_{24}, a_{42} \leq 0$ along with the tridiagonality condition ($a_{13} = a_{31} = a_{24} = a_{42} = 0$ when $|i - j| > 1$ and $|i - j| < n - 1$) give exactly the M_2^n conditions. The corner entries a_{14} and a_{41} must satisfy $a_{14}, a_{41} \leq 0$.

6.4 The General Case: $k \in [2, n - 2]$

Definition 6.3 (Definition 5 in the paper). For $k \in [2, n-2]$, M_k^n is the set of matrices $A \in \mathbb{R}^{n\times n}$ satisfying:

(a) $(-1)^{k-1}a_{1n}, (-1)^{k-1}a_{n1} \geq 0$;
(b) $a_{ij} \geq 0$ for all i, j with $|i - j| = 1$;
(c) $a_{ij} = 0$ for all i, j with $1 < |i - j| < n - 1$.

What this means: The matrix is still tridiagonal with nonneg. super/sub-diagonals, but the sign of the corner entries depends on k:

- For k even: $a_{1n}, a_{n1} \leq 0$ (same as M_2^n).
- For k odd: $a_{1n}, a_{n1} \geq 0$.

Sign Pattern of M_3^5

For $n = 5$, $k = 3$ (odd, so $(-1)^{k-1} = (-1)^2 = 1$, meaning $a_{1n}, a_{n1} \geq 0$):

$$\begin{bmatrix} * & \geq 0 & 0 & 0 & \geq 0 \\ \geq 0 & * & \geq 0 & 0 & 0 \\ 0 & \geq 0 & * & \geq 0 & 0 \\ 0 & 0 & \geq 0 & * & \geq 0 \\ \geq 0 & 0 & 0 & \geq 0 & * \end{bmatrix}.$$

Compare with M_2^5 above: the only difference is the sign of the corner entries a_{15} and a_{51}.

Theorem 6.4 (Theorem 8 in the paper)**.** *For any $k \in [2, n-2]$ the matrix $A^{[k]}$ is Metzler iff $A \in M_k^n$.*

Combined with Theorem 4, this gives: The LTV (1) is k-positive on (a, b) iff $A(s) \in M_k^n$ for almost all $s \in (a, b)$.

Corollary 6.5 (Corollary 3 in the paper)**.** *If there exist $i, j \in \{1, \ldots, n-2\}$ with i even and j odd such that the system is both i-positive and j-positive, then the system is k-positive for all $k \in [1, n]$.*

Why: i-positivity with i even forces $a_{1n}, a_{n1} \leq 0$. j-positivity with j odd forces $a_{1n}, a_{n1} \geq 0$. Together, $a_{1n} = a_{n1} = 0$, and A is tridiagonal Metzler, which means $A^{[k]}$ is Metzler for every k.

Even vs. Odd k and the Corner Entries

The sign of the corner entries a_{1n} and a_{n1} is the *only* difference between M_k^n for different values of k (with $k \in [2, n-2]$). Even k requires $a_{1n}, a_{n1} \leq 0$; odd k requires ≥ 0. This means a system can be k-positive for all even k but not for any odd k (or vice versa), depending solely on the signs of these two corner entries.

7 Geometrical Structure of the Invariant Sets

7.1 P_-^k Is a Cone (But Not Convex!)

The invariant sets P_-^k and P_+^k are **cones**: if $x \in P_-^k$, then $\alpha x \in P_-^k$ for all $\alpha \in \mathbb{R}$, since scaling a vector does not change its sign pattern.

P_-^k Is NOT Convex for $k \geq 2$

For $k \geq 2$, the sets P_-^k are *not* convex.
Consider $n = 2$: $x = [1,\ 1]'$ and $y = [-1,\ -1]'$ both have $s^-(x) = s^-(y) = 0$, so both are in P_+^1. But $\frac{1}{2}(x + y) = [0,\ 0]' \notin P_+^1$ (the zero vector is excluded from P_+^k for $k < n$).
For $n = 3$: $x = [1,\ -1,\ 0]'$ and $y = [0,\ -1,\ 1]'$ both satisfy $s^-(x) = s^-(y) = 1$, so both are in P_-^2. But $\frac{1}{2}(x + y) = [1/2,\ -1,\ 1/2]'$ has $s^- = 2$, so $\frac{1}{2}(x + y) \notin P_-^2$.
This means k-positive systems are *not* monotone in the usual (partial-order) sense, since monotonicity requires a convex cone.

7.2 P_-^k Is a k-Solid Cone

Despite not being convex, P_-^k has useful structure. Recall that a set $C \subseteq \mathbb{R}^n$ is a **cone of rank k** if:

1. C is closed,
2. $x \in C \implies \alpha x \in C$ for all $\alpha \in \mathbb{R}$,
3. C contains a linear subspace of dimension k and no linear subspace of higher dimension.

A cone C of rank k is **solid** if its interior is nonempty, and **k-solid** if there is a linear subspace W of dimension k such that $W \setminus \{0\} \subseteq \text{int}(C)$.

Lemma 7.1 (Lemma 5 in the paper). *For any $k \in [1, n-1]$, P_-^k is a k-solid cone, and its complement $(P_-^k)^c := \text{clos}(\mathbb{R}^n \setminus P_-^k)$ is an $(n-k)$-solid cone.*

Why this matters: The fact that P_-^k is k-solid means it contains a k-dimensional linear subspace W. For a k-positive system, trajectories starting in P_-^k can be projected onto this k-dimensional subspace, and under certain conditions (strong k-positivity), this projection is one-to-one. This effectively reduces the dynamics to a k-dimensional system, which is much easier to analyze.

Dimension Reduction Through Invariant Cones

The k-solid structure of P_-^k provides a "dimensional bottleneck": trajectories confined to P_-^k can be faithfully represented in k dimensions. For $k = 1$, this means trajectories in P_-^1 project to a line (1-D), leading to convergence to an equilibrium. For $k = 2$, trajectories in P_-^2 project to a plane (2-D), leading to the Poincaré–Bendixson property (convergence to an equilibrium or a closed orbit).

7.3 P_-^k as a Union of Convex Cones

Although P_-^k itself is not convex, it can be decomposed into a union of convex cones. For any $k \in [1, n]$, define

$$Q_-^k := \{z \in \mathbb{R}^n : s^-(z) = k - 1\}.$$

The following result gives a decomposition of P_-^s as a union of convex cones:

Proposition 7.2 (Proposition 9 in the paper). *For any $s \in [1, n]$ we have*

$$P_-^s = \bigcup_{k=1}^{s} Q_-^k,$$

where

$$Q_-^k = \bigcup_{i=1}^{\binom{n-1}{k-1}} \left(C_-^k(v^i) \cup (-C_-^k(v^i))\right),$$

and v^i, $i \in [1, \binom{n-1}{k-1}]$, are all the different vectors v satisfying $1 \le v_1 < v_2 < \cdots < v_k = n$.

Each Q_-^k is thus further decomposed into convex cones $C_-^k(v)$ and $-C_-^k(v)$, where the "pattern vector" $v = [v_1, \ldots, v_k]'$ specifies the positions of the sign changes.

Definition 7.1 (Definition 6 in the paper). For a vector $v = [v_1, \ldots, v_k]'$ with integer entries satisfying $1 \le v_1 < v_2 < \cdots < v_k = n$, let $C_-^k(v) \subseteq \mathbb{R}^n$ be the set of all $y \in \mathbb{R}^n$ satisfying:

- $y_1, \ldots, y_{v_1} \ge 0$, with at least one of these entries positive;
- $y_{v_1+1} < 0$, and $y_{v_1+2}, \ldots, y_{v_2} \le 0$;
- $y_{v_2+1} > 0$, and $y_{v_2+2}, \ldots, y_{v_3} \ge 0$; and so on until
- $(-1)^{k-1} y_{v_{k-1}+1} > 0$, and $(-1)^{k-1} y_{v_{k-1}+2}, \ldots, (-1)^{k-1} y_{v_k} \ge 0$ (recall that $v_k = n$).

What this means: Each $C^k_-(v)$ is an "orthant-like" set where the sign pattern is prescribed by v: entries in the first block are nonneg., entries in the second block are nonpos., and so on, alternating. The vector v tells you where the transitions between blocks occur.

Decomposition for n

With $v = [2,\ 3,\ 4]'$:

$$C^3_-(v) = \{y \in \mathbb{R}^4 : y_1, y_2 \geq 0,\ \max(y_1, y_2) > 0,\ y_3 < 0,\ y_4 > 0\}.$$

Note that $C^k_-(v)$ is a convex cone (it is an orthant in $\mathbb{R}^n$).
Proposition 9 shows $P^s_- = \bigcup_{k=1}^s Q^k_-$, and each $Q^k_- = \bigcup_i \big(C^k_-(v^i) \cup (-C^k_-(v^i))\big)$.

8 Applications to Nonlinear Systems

8.1 From Linear to Nonlinear: The Variational System

The paper applies the theory of k-positive linear systems to *nonlinear* systems through the **variational system**. Consider the time-varying nonlinear system:

$$\dot{x}(t) = f(t, x(t)), \tag{30}$$

with trajectories evolving on a convex state-space $\Omega \subseteq \mathbb{R}^n$.

For two initial conditions $p, q \in \Omega$, let $z(t) := x(t, p) - x(t, q)$ denote the difference between the two resulting trajectories. Then:

$$\dot{z}(t) = A^{pq}(t)\, z(t), \tag{31}$$

where

$$A^{pq}(t) := \int_0^1 J(t, r\, x(t, p) + (1 - r)\, x(t, q))\, \mathrm{d}r, \tag{32}$$

and $J(t, x) := \frac{\partial}{\partial x} f(t, x)$ is the Jacobian.

What this says in plain English: The difference $z(t)$ between any two trajectories satisfies a linear time-varying system whose matrix $A^{pq}(t)$ is an "average Jacobian" along the line segment between the two trajectories at time t. This LTV is called the **variational system**.

The Variational System Bridge

To study a nonlinear system using k-positivity theory:

1. Compute the Jacobian $J(t, x) = \frac{\partial f}{\partial x}$.
2. Check if $J(t, x) \in M^n_k$ for all $x \in \Omega$ and relevant t.
3. Since M^n_k is closed under convex combinations (it is defined by linear inequalities), $A^{pq}(t) \in M^n_k$ for all p, q, and the variational system is k-positive.

Definition 8.1 (Definition 7 in the paper)**.** The nonlinear system (30) is [strongly] k-cooperative if the variational system (31) is [strongly] k-positive for all $p, q \in \Omega$.

Note the terminology: *k-positivity* refers to linear systems; *k-cooperativity* refers to nonlinear systems (via their variational system). A 1-cooperative system is a standard cooperative system.

Corollary 8.1 (Corollary 4 in the paper)**.** *If $J(t, z) \in M^n_k$ for almost all $t \in (a, b)$ and all $z \in \Omega$, then (30) is k-cooperative on (a, b). If additionally $J(t, z)$ is irreducible at some τ^+ in every subinterval, then (30) is strongly k-cooperative.*

8.2 Example: Cyclic Feedback Systems

A nonlinear system of the form

$$\dot{x}_1 = f_1(x_1, x_n), \quad \dot{x}_i = f_i(x_{i-1}, x_i, x_{i+1}),\ i = 2, \dots, n-1, \quad \dot{x}_n = f_n(x_{n-1}, x_n), \tag{33}$$

is a **cyclic feedback system**: each variable depends only on itself and its nearest neighbors in a chain, with a "feedback" connection from x_n back to x_1. The Jacobian has the form:

$$J(x) = \begin{bmatrix} * & 0 & \cdots & 0 & \operatorname{sgn}(\delta_1) \\ >0 & * & >0 & \cdots & 0 \\ 0 & >0 & * & \ddots & \vdots \\ \vdots & & \ddots & & >0 \\ 0 & 0 & \cdots & >0 & * \end{bmatrix}$$

where the sign of the $(1, n)$ entry depends on a parameter $\delta_1 \in \{-1, 1\}$.

- If $\delta_1 = 1$: $J(x)$ is Metzler $\implies$ strongly 1-cooperative (cooperative).
- If $\delta_1 = -1$: $J(x) \in M_2^n \implies$ strongly 2-cooperative.

8.3 Example: Systems with Scalar Nonlinearities

Consider the system:

$$\dot{x}(t) = C(t) \begin{bmatrix} f_1(x_1(t)) \\ f_2(x_2(t)) \\ \vdots \\ f_n(x_n(t)) \end{bmatrix}, \tag{34}$$

where each $f_i : \mathbb{R} \to \mathbb{R}$ is C^1 and $C : (a, b) \to \mathbb{R}^{n \times n}$. The Jacobian is:

$$J(t, x) = C(t)\ \operatorname{diag}(f_1'(x_1), \dots, f_n'(x_n)). \tag{35}$$

If $f_i'(z) > 0$ for all z and all i (each f_i is strictly increasing), then substituting (35) into the integral (32) gives

$$A^{pq}(t) = C(t)\ \operatorname{diag}(g_1(p_1, q_1), \dots, g_n(p_n, q_n)),$$

where each $g_i(p_i, q_i) > 0$ (by compactness of Ω, there exists $\delta > 0$ such that $g_i \geq \delta$ for all i and all $p, q \in \Omega$). Hence every entry of $A^{pq}(t)$ satisfies $a_{ij}(t) = c_{ij}(t) \cdot g_j(p_j, q_j)$ with $g_j > 0$, so the sign of a_{ij} equals the sign of c_{ij}. Thus $A^{pq}(t) \in M_k^n$ iff $C(t) \in M_k^n$.

9 Invariant Sets of Nonlinear k-Cooperative Systems

9.1 Invariance of P_-^k

Proposition 9.1 (Proposition 10 in the paper)**.** *If (30) is k-cooperative, then for any $p, q \in \Omega$:*

$$p - q \in P_-^k \implies x(t, p) - x(t, q) \in P_-^k \text{ for all } t \geq 0. \tag{37}$$

If furthermore $0 \in \Omega$ is an equilibrium ($f(t, 0) = 0$), then:

$$p \in P_-^k \implies x(t, p) \in P_-^k \text{ for all } t \geq 0. \tag{38}$$

What (37) says: If two initial conditions p and q differ by a vector with at most $k-1$ sign variations, then the corresponding trajectories *at any future time* still differ by a vector with at most $k-1$ sign variations.

What (38) says: If the origin is an equilibrium, then P_-^k itself is an invariant set of the nonlinear dynamics. Starting in P_-^k means staying in P_-^k.

Explicit Invariant Sets for Nonlinear Systems

Proposition 10, combined with the decomposition $P^k_- = \bigcup C^k_-(v^i) \cup (-C^k_-(v^i))$, provides *explicit, analytically described* invariant sets for the nonlinear system. This is remarkable because finding invariant sets for nonlinear systems is generally very difficult.
For a 2-cooperative system with equilibrium at the origin, the invariant set P^2_- consists of all vectors with at most one sign change. In $\mathbb{R}^3$, this is the union of 4 "wedge-shaped" convex cones (one for each sign pattern like $[+,+,-]$, $[+,-,-]$, and their negatives).

If the system is *strongly* k-cooperative, the invariance strengthens:

$$p - q \in P^k_- \setminus \{0\} \implies x(t,p) - x(t,q) \in P^k_+ \text{ for all } t > 0.$$

The difference is mapped into the *interior* of P^k_-, providing a strict version of the invariance.

10 The Poincaré–Bendixson Property for 2-Cooperative Systems

10.1 Background: The Classical Poincaré–Bendixson Theorem

The **Poincaré–Bendixson theorem** is one of the most important results in the qualitative theory of differential equations. In its classical form (for 2-dimensional systems), it states:

Theorem 10.1 (Classical Poincaré–Bendixson)**.** *Consider $\dot{x} = f(x)$ with $x \in \mathbb{R}^2$. If a trajectory has a compact omega-limit set that does not contain any equilibrium, then the omega-limit set is a closed orbit (periodic solution).*

What this means: In 2D, the only possible long-term behaviors of a bounded trajectory are: convergence to an equilibrium, convergence to a periodic orbit, or a heteroclinic/homoclinic connection between equilibria. Chaos is impossible in 2D!

The Poincaré–Bendixson theorem fails in dimensions $n \geq 3$: bounded trajectories in 3D can exhibit chaotic behavior (e.g., the Lorenz attractor). However, for systems with special structure, the theorem can be recovered.

Why 2-Cooperativity Enables Poincaré–Bendixson

A strongly 2-cooperative system has P^2_- as an invariant 2-solid cone. Trajectories in P^2_- can be projected (one-to-one) onto a 2-dimensional subspace. Since the projection is one-to-one, the projected dynamics behave like a 2D system, and the classical Poincaré–Bendixson theorem applies to the projection. Pulling back to the original space gives the result for the n-dimensional system.

10.2 The Paper's Result

Remark 10.2 (Remark 7 in the paper). If the nonlinear system $\dot{x} = f(x)$ is 2-cooperative (i.e., $J(x) \in M^n_2$ for all x), then $J(x) \in M^n_i$ for all even i (because M^n_2 and M^n_i differ only in the corner entries, and M^n_2 requires $a_{1n}, a_{n1} \leq 0$, which also satisfies even-k requirements). Thus, the system is in fact $(2i)$-cooperative for all $i \geq 1$, and strongly 2-cooperative implies strongly $(2i)$-cooperative for all $i \geq 1$.

The main result of the paper regarding the Poincaré–Bendixson property is:

Theorem 10.3 (Theorem 12 in the paper)**.** *Suppose that $\dot{x} = f(x)$ is strongly 2-cooperative. Let $x(t, x_0)$ be a solution with a compact omega-limit set $\omega(x_0)$. If $\omega(x_0)$ does not include an equilibrium, then it is a closed orbit.*

What this says in plain English: For strongly 2-cooperative systems (which includes all cyclic feedback systems with negative feedback), any bounded trajectory either converges to an equilibrium or converges to a periodic orbit. Chaotic behavior is ruled out, regardless of the dimension n.

Why this is stronger than previous results: An earlier result by Sanchez (2009) established a similar Poincaré–Bendixson property, but only for *pseudo-ordered* solutions (trajectories whose velocity vector eventually enters P^2_-). Theorem 12 applies to *any* solution with a compact omega-limit set, without requiring pseudo-ordering. The key to this strengthening is the explicit characterization of P^2_- as a union of convex cones, combined with the nested structure of the invariant sets.

10.3 Proof Strategy (Simplified)

The proof of Theorem 12 proceeds in several steps:

1. **Lemma 6:** For a closed orbit γ of a strongly 2-cooperative system, any two points $p, q \in \gamma$ satisfy $p - q \in P^k_+$ for *every even* $k \geq 2$, or $p - q \notin P^k_-$ for any even k. This uses Remark 7 (2-cooperative $\implies$ $(2i)$-cooperative).
2. **Lemma 7:** For a closed orbit of a strongly 2-cooperative system, there exists an odd integer $\ell \geq 1$ such that for any two times $0 < t_2 - t_1 < T$ (the period):
$$\ell - 1 \leq s^-(x(t_2, x_0) - x(t_1, x_0)) \leq s^+(x(t_2, x_0) - x(t_1, x_0)) \leq \ell.$$
3. **Lemma 8:** The orthogonal projection of the closed orbit onto the 2-dimensional subspace $W^{1n} := \text{span}\{e^1, e^n\}$ is one-to-one. This is proved by contradiction: if two distinct points on the orbit project to the same point, the difference $p - q$ has $p_1 - q_1 = p_n - q_n = 0$, which forces $s^+(p - q) \geq 2 + s^-(p - q)$, contradicting Lemma 7.
4. **Conclusion:** Since the projection is one-to-one, the standard Poincaré–Bendixson theorem (from Sanchez, 2009) applies, and the omega-limit set is a closed orbit.

11 Connections and Special Cases

11.1 Totally Positive Differential Systems (TPDS)

A **totally positive differential system** (TPDS), introduced by Schwarz (1970), is an LTV $\dot{x} = A(t)x$ whose transition matrix is TP for all $t > t_0$. Schwarz showed that this holds iff $A \in \mathbb{M}^+$, where $\mathbb{M}^+$ is the set of irreducible tridiagonal matrices with positive off-diagonal entries.

In the language of this paper, a TPDS is a system that is strongly k-positive for *all* $k \in [1, n]$ simultaneously. The paper's framework recovers and generalizes this: one can be strongly k-positive for some k without being strongly j-positive for other j.

11.2 Cooperative and Competitive Systems

The classical theory of cooperative dynamical systems (Smith, 1995) studies systems where $J(x)$ is Metzler (1-positive). The theory of competitive systems studies systems where $(-J(x))$ is Metzler ($(n-1)$-positive, by Lemma 4). The paper's framework of k-positivity provides a smooth interpolation between these two extremes:

k	System type	Invariant set
$k = 1$	Cooperative	$P_-^1 = \mathbb{R}_+^n \cup \mathbb{R}_-^n$ (nonneg./nonpos. orthants)
$k = 2$	2-cooperative	P_-^2 (at most 1 sign change)
$\vdots$	$\vdots$	$\vdots$
$k = n - 1$	Competitive	P_-^{n-1} (at most $n - 2$ sign changes)
$k = n$	Any system	$P_-^n = \mathbb{R}^n$ (no restriction)

11.3 The Abel–Jacobi–Liouville Identity

For $k = n$, the compound matrix $A^{[n]}$ is 1×1 and equals $\mathrm{tr}(A)$. The dynamics (13) become:

$$\frac{d}{dt}(\det Y(t)) = \mathrm{tr}(A(t)) \det Y(t),$$

which is the classical **Abel–Jacobi–Liouville identity**. This identity describes how the volume element (determinant) of a fundamental matrix evolves. The paper's framework thus includes this classical result as the special case $k = n$.

For $k = 1$, $A^{[1]} = A$ and (13) is just the original system $\dot{Y} = AY$. So the framework interpolates between the original system ($k = 1$) and the scalar determinant equation ($k = n$).

12 Summary of Notation

For quick reference, here is a table of all major symbols used in the paper:

Symbol	Type / Dimensions	Meaning
$x \in \mathbb{R}^n$	Vector	State vector of the dynamical system
$A(\tau) \in \mathbb{R}^{n\times n}$	Matrix	System matrix of the LTV at time τ
$\Phi(t, t_0) \in \mathbb{R}^{n\times n}$	Matrix	Transition (fundamental) matrix
$\mathbb{R}^n_+$	Set	Nonnegative orthant $\{x : x_i \geq 0 \ \forall i\}$
$\sigma(y)$	Scalar (integer ≥ 0)	Number of sign variations (no zeros)
$s^-(y)$	Scalar (integer ≥ 0)	Min. sign variations (delete zeros)
$s^+(y)$	Scalar (integer ≥ 0)	Max. sign variations (worst-case zeros)
$\mathcal{V}$	Set	Vectors where $s^- = s^+$
D	Diagonal matrix	$\mathrm{diag}(1, -1, 1, \ldots, (-1)^{n-1})$
SR_k, SSR_k	Matrix property	Sign-regular / strictly sign-regular of order k
$A(\alpha\|\beta)$	Scalar	Minor of A with rows α, columns β
$A^{(k)}$	$\binom{n}{k} \times \binom{n}{k}$ matrix	kth multiplicative compound of A
$A^{[k]}$	$\binom{n}{k} \times \binom{n}{k}$ matrix	kth additive compound of A
P^k_-	Set (closed cone)	$\{z : s^-(z) \leq k-1\}$
P^k_+	Set (open cone)	$\{z : s^+(z) \leq k-1\} = \mathrm{int}(P^k_-)$
M^n_k	Matrix class	Matrices A such that $A^{[k]}$ is Metzler
$\mathbb{M}, \mathbb{M}^+$	Matrix classes	Tridiagonal Metzler / with positive off-diag.
Q^k_-	Set	$\{z : s^-(z) = k-1\}$ (exact $k{-}1$ sign changes)
$C^k_-(v)$	Convex cone	Orthant-like set with sign pattern given by v
$f(t, x)$	Vector field	Right-hand side of the nonlinear system
$J(t, x) \in \mathbb{R}^{n\times n}$	Matrix	Jacobian $\frac{\partial}{\partial x} f(t, x)$
$A^{pq}(t) \in \mathbb{R}^{n\times n}$	Matrix	Average Jacobian along segment $[q, p]$
$\omega(x_0)$	Set	Omega-limit set of trajectory from x_0
k-positive	System property	LTV maps P^k_- to itself
k-cooperative	System property	Nonlinear system with k-positive variational eq.

Main Results	Summary
Theorem 4	k-positive $\iff$ $A^{[k]}$ Metzler
Theorem 5	Sufficient condition for strong k-positivity
Theorem 6	Necessary and sufficient condition for strong k-positivity (continuous A)
Proposition 7	Decreasing staircase of sign variations
Theorem 8	$A^{[k]}$ Metzler $\iff$ $A \in M^n_k$
Corollary 2	k-positive $\iff$ $A(s) \in M^n_k$ for a.e. s
Lemma 4	$(n{-}1)$-positive = competitive (via coord. transform)
Lemma 5	P^k_- is a k-solid cone
Proposition 9	P^k_- is a union of convex cones
Proposition 10	Invariant sets for k-cooperative nonlinear systems
Theorem 12	Poincaré–Bendixson for strongly 2-cooperative systems